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General Yang–Baxterization of reflection equations and general K-matrices of A_{n-1} vertex models

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Abstract. General Yang–Baxterization of reflection equations for $\check{R}$ with two distinct eigenvalues is presented. This procedure is used to construct reflection K-matrices of reflection equations for the A_{n-1} vertex models. (Diagonal) K-matrices by de Vege and González-Ruiz for the A_{n-1} vertex model are reproduced. Furthermore, some new off-diagonal K-matrices for the A_{n-1} vertex model are obtained.

1. Introduction

It is well known that the lattice integrable models with periodic boundary conditions are described by the quantum Yang–Baxter equations. Recently, much attention has been paid to the lattice integrable models with non-periodic boundary conditions [1–5]. This kind of integrable models is described by the quantum $\check{R}$ matrix on the bulk and by the reflection K-matrices $K^{\pm}(x)$ on the left and the right boundaries. If we know the $\check{R}$ and the reflection K-matrices we can immediately derive the integrable Hamiltonians [1, 3, 9]. The quantum R-matrix satisfying the quantum Yang–Baxter equation has been extensively studied by many authors. Therefore, it is very important to study the reflection K-matrices for constructing new integrable models with non-periodic boundary conditions.

The reflection K-matrix $K^{-}(x)$ satisfies the *reflection equation* (RE) proposed by Cherednik and Sklyanin [1, 3]

$$\check{R}(xy^{-1})K_1^{-}(x)\check{R}(xy)K_1^{-}(y)=K_1^{-}(y)\check{R}(xy)K_1^{-}(x)\check{R}(xy^{-1}) \tag{1.1}$$

where $K_1^{-}(x)=K^{-}(x)\otimes 1$. The $K^{+}(x)$ can be obtained from $K^{-}(x)$ in terms of a simple transformation. Therefore, it is of significance to solve the RE (1.1).

De Vega and González-Ruiz [5] recently solved directly the RE for six- and eight-vertex models and the diagonal K-matrices of the A_{n-1} models. This method is more complicated in practical calculations because we have to solve the set of functional equations of the matrix elements of RE (1.1).

In this paper we shall propose a different method, called the Yang–Baxterization method, for finding K-matrices. The basic idea is to establish a procedure, called Yang–Baxterization, to convert the parameter-independent reflection K-matrices, which satisfy the parameter-independent RE

$$\check{R}K_1^-\check{R}K_1^- = K_1^-\check{R}K_1^-\check{R} \tag{1.2}$$

into the parameter-dependent K-matrices. It is easy to see that this method enjoys the following two merits:

(i) It simplifies the set of functional equations of the matrix elements of the parameter-dependent RE (1.1) to the algebraic equations of matrix elements of the parameter-independent RE (1.2). The latter has been extensively studied by Kulish *et al* [7]. We can therefore obtain a larger class of solutions to RE (1.1)

(ii) Using this method one can also obtain the algebraic solution of RE (1.1). Sklyanin and Kulish introduced the algebraic structures related to the parameter-independent RE (1.2), which are known as the reflection algebras. In terms of the Yang–Baxterization procedure these algebraic solutions convert to parameter-dependent algebraic solutions of RE (1.1).

The key point of our method is the Yang–Baxterization procedure. In section 2 we shall present a general Yang–Baxterization procedure for $\check{R}$ with two distinct eigenvalues. Note that in a previous letter [6] we have presented a Yang–Baxterization of the RE for $\check{R}$ with two distinct eigenvalues, which is in fact a special case of the results in this paper. Then using this procedure we investigate the reflection K-matrices for the A_{n-1} models. Results by de Vega and González-Ruiz are easily reproduced in section 3, and some new off-diagonal K-matrices of A_{n-1} vertex model are obtained in section 4.

2. General Yang–Baxterization of the RE

The $\check{R}$ having two distinct eigenvalues t_1, t_2 can be Yang–Baxterized as [8]

$$\check{R}(x) = (x - x^{-1})\check{R} - (t_1 + t_2)xI. \tag{2.1}$$

So what we need to do is to incorporate the parameter x into parameter-independent K-matrices such that they satisfy RE (1.1). Considering the fact that all the constant solutions for $\check{R}$ having two distinct eigenvalues derived by Kulish *et al* [7] satisfy a quadratic relation $(K^-)^2 + AK^- = CI$, where $A, C \in \mathbb{C}$[6], we can generally suppose that $K^-(x)$ depends only on the zeroth and first order of K^-

$$K^-(x) = K^- + f(x)I \tag{2.2}$$

where $f(x)$ is a function to be determined. Inserting (2.1) and (2.2) into (1.1), and using $\check{R}^2 = (t_1 + t_2)\check{R} - t_1t_2I$, we find that

$$[\check{R}, (K_1^-)^2 + D(x, y)K_1^-] = 0 \tag{2.3}$$

where

$$D(x, y) = \frac{f(y)(y^2 - y^{-2}) - f(x)(x^2 - x^{-2})}{y^2 - x^2}. \tag{2.4}$$

Since x, y are arbitrary parameters, the function $D(x, y)$ has to be a constant, which is denoted by A. Then we have

$$f(y)(y^2-y^{-2})-Ay^2=f(x)(x^2-x^{-2})-Ax^2 \tag{2.5}$$

which has to be a constant B from the arbitrariness of x, y. From this equation we obtain the final result

$$f(x)=\frac{B+Ax^2}{x^2-x^{-2}} \tag{2.6}$$

in which B is a free parameter and A is determined by

$$[\check{R}, (K_1^-)^2+AK_1^-]=0. \tag{2.7}$$

Let us consider a special case. If $B=A$, then $f(x)$ reduces to

$$f(x)=\frac{Ax}{x-x^{-1}} \tag{2.8}$$

which is just the result presented in [6] (for $K^-(x)$ up to $x-x^{-1}$).

We would like to note that the above presentation is also true for the algebraic K-matrices if $f(x)$ is a function of central elements of the reflection algebras (RA), namely A is a central element of the RA. This means that the algebraic solutions of $sl(2)_q$ (see next section) and $su(1,1)_q$ models can be Yang–Baxterized by means of the above procedure.

3. *K*-matrices of the six-vertex model

We now apply the above procedure to the construction of general K-matrices for the A_{n-1} models. We first consider the six-vertex model in this section. Results by de Vega and González-Luiz are reproduced.

The $\check{R}$ matrix for six-vertex models reads

$$\check{R}=\begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & \omega & \\ & & & q \end{pmatrix} \qquad \omega=q-q^{-1}. \tag{3.1}$$

From (2.1) (with $t_1=q$, $t_2=-q^{-1}$) we get its well known parameter-dependent form (up to $xq-(xq)^{-1}$)

$$\check{R}(x)=\begin{pmatrix} 1 & & & \\ & bx^{-1} & a & \\ & a & bx & \\ & & & 1 \end{pmatrix} \qquad \begin{cases} a=\dfrac{x-x^{-1}}{xq-(xq)^{-1}} \\ b=\dfrac{\omega}{xq-(xq)^{-1}}. \end{cases} \tag{3.2}$$

Letting

$$K^{-}=\begin{pmatrix}\gamma & \beta\\ \alpha & \delta\end{pmatrix} \tag{3.3}$$

we obtain the algebraic relations of the reflection algebra $\mathscr{A}_1$

$$\begin{aligned}&\delta\alpha=q^{-2}\alpha\delta \qquad [\delta,\gamma]=0 \qquad [\alpha,\gamma]=-q^{-1}\omega\delta\alpha\\ &\delta\beta=q^{2}\beta\delta \qquad [\alpha,\beta]=q^{-1}\omega(\delta\gamma-\delta^{2}) \qquad [\beta,\gamma]=q^{-1}\omega\beta\gamma.\end{aligned} \tag{3.4}$$

This algebra has two central elements

$$C_1=\delta+q^{2}\gamma \qquad C_2=\det_q K^{-}=\delta\gamma-q^{2}\beta\alpha,$$

satisfying the relation

$$(K^{-})^{2}-q^{-2}C_1K^{-}=-q^{-2}C_2. \tag{3.5}$$

This means that we can choose $A=-q^{-2}C_1$, and therefore the algebraic solution is Yang–Baxterized as

$$K^{-}(x)=K^{-}+\frac{B-(q^{-2}\delta+\gamma)x^{2}}{x^{2}-x^{-2}}I. \tag{3.6}$$

There are two constant solutions. Besides the identity solution, the other one reads

$$K^{-}=\begin{pmatrix}\gamma & \beta\\ \alpha & 0\end{pmatrix} \tag{3.7}$$

with three free parameters α, β, γ. Then we have (up to $x^{2}-x^{-2}$)

$$K^{-}(x)=\begin{pmatrix}B-\gamma x^{-2} & \beta(x^{2}-x^{-2})\\ \alpha(x^{2}-x^{-2}) & B-\gamma x^{2}\end{pmatrix}. \tag{3.8}$$

To produce the result in [5], we rearrange the parameters as $B=\gamma\exp(-2\xi)$, $x=\exp(\theta)$. Then equation (3.8) is rewritten as

$$K^{-}(\theta)=\begin{pmatrix}k\sinh(\xi-\theta)\mathrm{e}^{-\theta} & \mu\sinh(2\theta)\\ \eta\sinh(2\theta) & k\sinh(\xi+\theta)\mathrm{e}^{\theta}\end{pmatrix} \tag{3.9}$$

with $k=-2\gamma\mathrm{e}^{-\xi}$, $\mu=2\beta$, $\eta=2\alpha$. In comparison with the result in [5], there are two additional factors $\mathrm{e}^{\pm\theta}$ in the diagonal elements. This is because the $\check{R}$ matrix we use is different from that in [5] and these factors can be cancelled by a gauge transformation.

Here we would like to note that the $K^{-}(x)$ given above also includes the identity solution (up to any function $F(x)$ about x) by setting $\alpha=\beta=\gamma=0$.

4. K-matrices of A_{n-1} models

The diagonal K-matrix by de Vega and González-Ruiz for the A_{n-1} model can be easily derived from the following constant solution of RE (1.1)

$$K^{-}(l)_{ij}=\begin{cases}\delta_{ij} & \text{if } 1\leqslant i\leqslant l\\ 0 & \text{if } l+1\leqslant i\leqslant n\end{cases} \qquad (l=1,2,\ldots,n) \tag{4.1}$$

where the $\check{R}$ for the A_{n-1} model reads

$$\check{R}=q\sum_i e_{ii}\otimes e_{ii}+\sum_{i\neq j} e_{ji}\otimes e_{ij}+\omega\sum_{i<j} e_{jj}\otimes e_{ii}. \tag{4.2}$$

It is easy to see that $(K^-(l))^2-K^-(l)=0$ and $A=-1$. Therefore, its Yang–Baxterization is derived as

$$K^-(x,l)_{ij}=\begin{cases}\delta_{ij}(B-x^{-2}) & \text{if } 1\leqslant i\leqslant l\\ \delta_{ij}(B-x^2) & \text{if } l+1\leqslant i\leqslant n.\end{cases} \tag{4.3}$$

Redefining the parameters as in the A_1 case we get the diagonal K-matrix in [6] (up to $-2e^{-\xi}$)

$$K^-(\theta,l)_{ij}=\begin{cases}\delta_{ij}\sinh(\xi-\theta)e^{-\theta} & \text{if } 1\leqslant i\leqslant l\\ \delta_{ij}\sinh(\xi+\theta)e^{\theta} & \text{if } l+1\leqslant i\leqslant n\end{cases}. \tag{4.4}$$

We have seen that we can easily obtain the K-matrices by our Yang-Baxterization formulism. By means of this procedure solving the parameter-dependent RE reduces to solving the parameter-independent RE, in other words, solving the functional equations reduces to solving the algebraic equations. Therefore, we can deal with a larger class of K-matrices, not only the A_{n-1} model but also the $su_q(1,1)$ and eight-vertex models. In particular, we can also deal with the algebraic solutions as shown in the case of the six-vertex model presented in section 3. Now we present some off-diagonal K-matrices for A_{n-1} models. To use directly the constant solutions presented in [7], however, the $\check{R}$ we now use is a bit different from equation (4.2), namely the last term becomes $\omega\Sigma_{i>j}e_{jj}\otimes e_{ii}$.

For the A_2 model there are two constant solutions with non-vanishing determinants, one is the identity solution, the other one is

$$K^-=\begin{pmatrix} & & a_{13}\\ & a_{22} & \\ g_{31} & & a_{33}\end{pmatrix}$$

where $g_{31}=a_{22}(a_{22}-a_{33})/a_{13}$. It is easy to verify that

$$(K^-)^2-a_{33}K^-=a_{13}g_{31}I \qquad A=-a_{33}. \tag{4.5}$$

Therefore, they can be Yang–Baxterized as

$$K^-(x)=\begin{pmatrix}\dfrac{B-a_{33}x^2}{x^2-x^{-2}} & & a_{13}\\ & a_{22}+\dfrac{B-a_{33}x^2}{x^2-x^{-2}} & \\ g_{31} & & \dfrac{B-a_{33}x^{-2}}{x^2-x^{-2}}\end{pmatrix} \tag{4.6}$$

which includes five free parameters, x, a_{13}, a_{22}, a_{33} and B.

For the A_3 model, besides the unity solution, the following two constant solutions with non-vanishing determinants were presented in [7]:

$$K^-(0)=\begin{pmatrix} & & & a_{14} \\ & & a_{23} & \\ & a_{32} & a_{33} & \\ g_{41} & & & a_{33} \end{pmatrix} \qquad K^-(1)=\begin{pmatrix} & & & a_{14} \\ & a_{22} & & \\ & & a_{22} & \\ a_{41} & & & g_{44} \end{pmatrix}$$

where $g_{41}=a_{23}a_{32}/a_{14}$ and $g_{44}=a_{22}-(a_{14}a_{41})/a_{22}$. Both solutions can be combined as

$$K^-(\lambda)=\begin{pmatrix} & & & a_{14} \\ & \lambda a_{22} & (1-\lambda)a_{23} & \\ & (1-\lambda)a_{32} & \Lambda & \\ \Gamma & & & \Delta \end{pmatrix} \qquad \begin{aligned} \Lambda &= \lambda a_{22}+(1-\lambda)a_{33} \\ \Gamma &= \lambda a_{41}+(1-\lambda)g_{41} \\ \Delta &= (1-\lambda)a_{33}+\lambda g_{44} \end{aligned} \tag{4.7}$$

in terms of a discrete parameter $\lambda=0, 1$. One can prove that

$$[K^-(\lambda)]^2-\Delta K^-(\lambda)=[(1-\lambda)a_{23}a_{32}+\lambda a_{14}a_{41}]I \tag{4.8}$$

namely, $A=\Delta$. Then we can Yang–Baxterize the solution $K^-(\lambda)$ as

$$K^-(x, B, \lambda)=\begin{pmatrix} \Xi & & & a_{14} \\ & \Xi+\lambda a_{22} & (1-\lambda)a_{23} & \\ & (1-\lambda)a_{32} & \Xi+\Lambda & \\ \Gamma & & & \Xi+\Delta \end{pmatrix}$$

$$\Xi=\frac{B-\Delta x^2}{x^2-x^{-2}} \tag{4.9}$$

which includes eight continuous parameters, $x, B, a_{14}, a_{22}, a_{23}, a_{32}, a_{33}, a_{41}$, and a discrete parameter λ.

For $sl_q(n)$, there also exists an off-diagonal constant solution K^- with matrix elements

$$K^-_{ij}=\delta_{in+1-i}. \tag{4.10}$$

This solution satisfies $(K^-)^2=I$, then $A=0$. Therefore, its Yang–Baxterization is

$$K^-(x, B)_{ij}=\delta_{in+1-i}+\frac{B}{x^2-x^{-2}}\delta_{ij}. \tag{4.11}$$

5. Concluding remarks

We point out that in the above presentation only the A_1 case is the general K-matrix because it is derived from all the constant solutions. For the cases A_2 and A_3, we only considered the constant solutions with non-vanishing determinants (including the identity solution), and for the A_{n-1} case, we only present an example of off-diagonal K-matrix.

To obtain the general K-matrix we have to consider all the constant solutions. One first mixes up all the constant solutions in a general form in terms of a set of discrete parameters as shown in the A_3 case, then obtain its parameter-dependent form using the Yang–Baxterization procedure. Since we usually have more than one constant solution, the parameter-dependent K-matrix includes not only the continuous but also the discrete parameters.

For the A_{n-1} case we can add a set of parameters to the constant solutions through the parameter-dependent transformation (for details, see [7]). For instance, we can get a counter-constant solution involving $n/2$ or $(n-1)/2$ parameters from the solution (4.10). Therefore, some additional parameters will enter the general K-matrices.

The associated integrable Hamiltonians can be derived from

$$H=C\left\{\sum_{n=1}^{N-1} h_{nn+1}+\frac{1}{2}\dot{K}_1^-(0)+\frac{\mathrm{tr}_0[K_0^{+t}(q^{n/2})h_{N0}]}{\mathrm{tr}[K^+(q^{n/2})]}\right\}, \tag{5.1}$$

where C is an arbitrary constant and

$$h_{nn+1}=\dot{\check{R}}_{nn+1}(0). \tag{5.2}$$

However, since the $\check{R}$ matrix for the A_{n-1} model does not enjoy the P, T and the crossing symmetry, $K^+(x)$ is not equal to $K^-(x)$, but can be obtained from $K^-(x)$ by

$$K^+(x)=K^-(x^{-1}q^{n/2})^t M. \tag{5.3}$$

where

$$M_{ab}=\delta_{ab}q^{n-2a+1} \qquad 1\leqslant a,b\leqslant n. \tag{5.4}$$

Then the integrable Hamiltonians related to the boundary conditions presented above can be easily derived.

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